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Interface crack between two different viscoelastic media

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Abstract

The plane (including antiplane) problem of an interfacial crack between different viscoelastic (including viscoelastic and elastic) media is considered. By using the Laplace transform method, the viscoelastic problem is reduced to an associated elastic one. The corresponding elastic analysis results in the viscoelastic solutions in the transformed field. The crack tip fields and fracture parameters of the viscoelastic interface crack are derived through an approximate Laplace inverse transform method. As an example, numerical calculations for an interfacial crack between viscoelastic and elastic materials are carried out. It is shown that the simple formulae of the crack line (and tip) fields and fracture parameter (energy release rate) of the viscoelastic interface crack, derived by the approximate method, are quite accurate. When the bimaterial is subject to a remote uniform and constant tensile loading, the normal stress along the crack line (including tip) is almost time independent. In contrast, the relative crack surface displacements and crack energy release rate do change with time, and are very much dependent on the creep compliance of the viscoelastic material. The tendencies of the crack advancing along the interface, and kinking out of the interface, are estimated and discussed.

Keywords: Interface crack; Viscoelastic; Fracture; Composites; Laplace transform

1. Introduction

Matrices of fiber-reinforced composites, or the cohesive layers of bonding laminates may creep, exhibiting viscoelastic behavior, under some service conditions. The interfacial fracture problem plays an important role in the failure analysis of composite material, and thus, has been receiving considerable attention. In contrast to the elastic interfacial crack problem, and the fracture of homogeneous viscoelastic materials, there are only a few works reported in the literature which treat interfacial crack problems in two different viscoelastic or viscoelastic/elastic medium. Sills and Benveniste (1981, 1983) determined the stress intensity factor of a steady propagating interface crack between two different viscoelastic half-planes. Atkinson and Bourner (1989) investigated the problem of a semi-infinite crack meeting a plane welded

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interface between two dissimilar viscoelastic materials Atkinson and Chen (1996) and Chang (1999) have studied the problem of a cracked layer bonded to a viscoelastic substrate. To simplify the analyses, all these works dealt with the anti-plane crack (mode III) problems only.

For a viscoelastic interfacial crack problem, in principle, it may be possible to use the so-called "correspondence principle". The principle enables the viscoelastic problems to be reduced to mathematically equivalent elastic ones (the associated elastic problems). After using an elastic analysis method to obtain the solution of the associated elastic problem, the viscoelastic solution can be obtained by the Laplace inverse transform method. Solutions of the elastic interfacial crack problems have indicated that due to the mismatch in bi-material moduli, except for the anti-plane crack case, the interface crack tip fields usually have an "oscillatory singularity", which introduce some complications that are not present in the fracture mechanics of homogeneous solids. This material dependent singularity then would evolve with time in the viscoelastic solution. This complex time-dependent singularity also occurs when a crack tip approaches the interface between two viscoelastic materials (Atkinson and Bourner, 1989). This complexity makes the exact Laplace inverse transform practically impossible. Due to this difficulty, except for the anti-plane case, the plane interfacial viscoelastic crack problems has not as yet been analyzed, though they represent a more realistic application background. It was mentioned earlier that a characteristic of an interfacial crack is its oscillatory behavior at the crack tips. To resolve this, a contact zone model at the crack tips has been employed. However, if this type of model is used for a viscoelastic problem, then the contact zone would also evolve with time. It is to be noted that for a mode I dominated loading, the oscillatory zone (or contact zone) is confined to a small range; consequently, the classical solution tends to be a reasonable approximation. Therefore, in this paper we will adopt the classical time-independent "oscillatory singularity" solution. Note, however, that the relative crack surface displacements and energy release rate would be time dependent. With regards to the correspondence principle and viscoelastic problems, the readers are encouraged to consult works by Schapery (1961, 1967) and Lee (1962) among others. The elastic interfacial crack problem has been summarized in a review article by Hutchinson and Suo (1991).

The plane (including the simple anti-plane) problem of an interface crack between two different viscoelastic (including viscoelastic and elastic) media is analyzed here. First, by using the Laplace transform method, the viscoelastic problem is reduced to an associated elastic one. Using the elastic methodology, the stress and strain fields of the viscoelastic problem are obtained in the transformed field. To overcome the difficulty involved in the Laplace inverse transform of the associated elastic solution, an approximate method is proposed. This then enables solutions along interface crack line (and at tip) and fracture parameters (such as energy release rate) to be obtained analytically. Finally, an interfacial crack between a viscoelastic and an elastic materials is analyzed to show the crack line/tip fields and energy release rate which vary with time. The numerical results show that the simple formulae obtained by the approximate method are quite accurate when compared with the numerical method of Laplace inverse transform. The viscoelastic effects on fracture parameters of an interfacial crack are discussed.

2. Constitutive equations

For isotropic linear elastic materials, the stress–strain relation can be expressed in deviatoric form (for the convenience of comparison with viscoelastic one), as

$$s_{ij} = 2\mu e_{ij}, \qquad \sigma_{kk} = 3K\varepsilon_{kk} \tag{1}$$

with s_{ij} and e_{ij} are deviatoric stress and strain tensors, μ and K are shear and bulk moduli, respectively.

For isotropic linear viscoelastic materials, the time dependent stress–strain constitutive equations can be expressed either in an integral form,

$$s_{ij}(t) = \int_{-\infty}^{t} G_1(t-\tau)\,\mathrm{d}e_{ij}(\tau), \qquad \sigma_{kk}(t) = \int_{-\infty}^{t} G_2(t-\tau)\,\mathrm{d}\varepsilon_{kk}(\tau) \tag{2a}$$

or in a differential form as,

$$P_1(D)s_{ij} = Q_1(D)e_{ij}, \qquad P_2(D)\sigma_{kk} = Q_2(D)\varepsilon_{kk} \tag{2b}$$

In the above $G_1(t)$, $G_2(t)$ are shear and bulk stress relaxation functions, respectively, P_1, Q_1, P_2 and Q_2 represent polynomials of the time derivative operator $D = \partial/\partial t$.

Taking the Laplace transform of Eqs. (2a) and (2b), and assuming that prior to time $t = 0$ all stresses and strains are zero, we obtain

$$\hat{s}_{ij}(p) = p\hat{G}_1(p)\hat{e}_{ij}(p), \qquad \hat{\sigma}_{kk}(p) = p\hat{G}_2(p)\hat{\varepsilon}_{kk}(p) \tag{3a}$$

$$P_1(p)\hat{s}_{ij}(p) = Q_1(p)\hat{e}_{ij}(p), \qquad P_2(p)\hat{\sigma}_{kk}(p) = Q_2(p)\hat{\varepsilon}_{kk}(p) \tag{3b}$$

where the notation $\hat{f}(p)$ denotes the Laplace transform of $f(t)$, that is

$$\hat{f}(p) = \mathscr{L}[f(t)] = \int_{0}^{\infty} f(t)\exp(-pt)\,\mathrm{d}t$$

And $\mathscr{L}^{-1}$ expresses Laplace inverse transform, that is $\mathscr{L}^{-1}[\hat{f}(p)] = f(t)$.

By defining new moduli, the equivalent shear and bulk moduli $\tilde{\mu}$ and $\tilde{K}$, as

$$2\tilde{\mu} = p\hat{G}_1(p) = Q_1(p)/P_1(p), \qquad 3\tilde{K} = p\hat{G}_2(p) = Q_2(p)/P_2(p), \tag{4}$$

Eqs. (3a) and (3b) are reduced to the following forms:

$$\hat{s}_{ij}(p) = 2\tilde{\mu}\hat{e}_{ij}(p), \qquad \hat{\sigma}_{kk}(p) = 3\tilde{K}\hat{\varepsilon}_{kk}(p) \tag{5}$$

Comparing Eq. (5) with the constitutive Eq. (1) for elastic materials, it can be seen that the constitutive equations of viscoelastic materials, in the Laplace transformed field, are similar to those of elastic materials. In addition, other basic equations including stress equilibrium and compatibility equations, are all similar to the corresponding elastic ones. Consequently, viscoelastic medium can be treated as an elastic one in the transformed field. The viscoelastic solutions can then be obtained from the associated elastic solutions by using the Laplace inverse transform.

3. Interfacial crack fields

The quasi-statically loaded interface crack problem we will consider is shown in Fig. 1, in which the medium 1 and 2 are different viscoelastic materials, or viscoelastic/elastic bi-materials. In the following formulation, the two materials are assumed to be two different viscoelastic materials. When one (or even both) material is elastic, the solution will reduce to viscoelastic/elastic (or bi-elastic) interface crack problem. And when the two materials are the same, it will further reduce to crack problems in homogeneous viscoelastic (or elastic) materials.

Since in the transformed field, the viscoelastic problem can be reduced to the associated elastic problem, the method and results of the elastic interfacial crack problem will be adopted directly in the following.

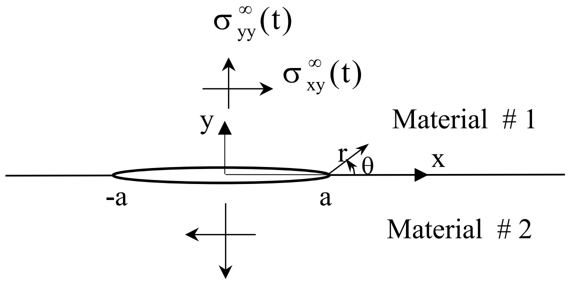

Fig. 1. Interface crack configuration.

4. Full field solution of an interface crack

For an isotropic elastic body under plane deformation, it is well known that stresses and displacements can be represented by two complex potentials $\varphi(z)$ and $\psi(z)$, or another pair of potentials $\Phi(z) = \varphi'(z)$ and $\Omega(z) = [z\varphi'(z) + \psi(z)]'$, see Muskhelishvili (1953) or Suo (1989). In the Laplace transformed field, the relations are

$$\begin{aligned} &\hat{\sigma}_{xx} + \hat{\sigma}_{yy} = 2\left[\hat{\Phi}(z) + \overline{\hat{\Phi}(z)}\right] \\ &\hat{\sigma}_{yy} - \mathrm{i}\hat{\sigma}_{xy} = \hat{\Phi}(z) + \overline{\hat{\Omega}(z)} + (z - \bar{z})\overline{\hat{\Phi}'(z)} \\ &2\tilde{\mu}\frac{\partial}{\partial x}(\hat{u}_x + \mathrm{i}\hat{u}_y) = \tilde{\kappa}\hat{\Phi}(z) - \overline{\hat{\Omega}(z)} - (z - \bar{z})\overline{\hat{\Phi}'(z)} \end{aligned} \tag{6}$$

where $\tilde{\kappa} = 3 - 4\tilde{\nu}$ for the plane strain and $\tilde{\kappa} = (3 - \tilde{\nu})/(1 + \tilde{\nu})$ for the plane stress, and the equivalent Poisson's ratio $\tilde{\nu} = 0.5(3\tilde{K} - 2\tilde{\mu})/(3\tilde{K} + \tilde{\mu})$, $\tilde{\mu}$ and $\tilde{K}$ are equivalent shear and bulk moduli defined in Eq. (4) for a viscoelastic medium, and $\tilde{\mu} = \mu$ and $\tilde{\kappa} = \kappa$ for an elastic medium.

Note the difference between the equivalent modulus $\tilde{f}$ and Laplace transform $\hat{f}(p) = \mathscr{L}[f(t)]$. The relation is $\tilde{f} = p\hat{f}$. For an elastic material, f is constant, $\hat{f} = \mathscr{L}[f] = f/p$, then $\tilde{f} = p\hat{f} = f$.

For the specified boundary conditions, the interfacial crack problem between two viscoelastic materials can be solved, similar to the corresponding elastic problem. The results are given directly according to the corresponding elastic problem. The complex potentials in the two media can be expressed as (Suo, 1989; Zhang and Li, 1992):

$$\begin{aligned} &\hat{\Phi}_1(z) = (1 + \tilde{\beta})\hat{F}(z), \qquad \hat{\Omega}_1(z) = (1 - \tilde{\beta})\overline{\hat{F}}(z) \\ &\hat{\Phi}_2(z) = (1 - \tilde{\beta})\hat{F}(z), \qquad \hat{\Omega}_2(z) = (1 + \tilde{\beta})\overline{\hat{F}}(z) \end{aligned} \tag{7}$$

where subscripts 1 and 2 refer to the two materials, $\tilde{\beta}$ corresponds to one of the Dunders' elastic mismatch parameters: $\tilde{\beta} = (\tilde{\mu}_1(\tilde{\kappa}_2 - 1) - \tilde{\mu}_2(\tilde{\kappa}_1 - 1))/(\tilde{\mu}_1(\tilde{\kappa}_2 + 1) + \tilde{\mu}_2(\tilde{\kappa}_1 + 1))$ and $\hat{F}(z)$ is an analytical function in the whole plane except on the crack line and depends on boundary conditions.

For a finite interface crack of length $2a$, under remote uniform tension and shear loading, $\sigma_{yy}^{\infty}(t)$ and $\sigma_{xy}^{\infty}(t)$, the function $\hat{F}(z)$ is taken to be (Rice and Sih, 1965; Zhang and Lee, 1993):

$$\hat{F}(z) = \frac{1}{2}\frac{\hat{\sigma}_{yy}^{\infty} - \mathrm{i}\hat{\sigma}_{xy}^{\infty}}{\sqrt{z^2 - a^2}}\left(\frac{z + a}{z - a}\right)^{\mathrm{i}\tilde{\varepsilon}}(z - 2\mathrm{i}\tilde{\varepsilon}a) \tag{8}$$

where $\tilde{\varepsilon}$ is the bi-material oscillatory singularity index,

$$\tilde{\varepsilon} = \frac{1}{2\pi} \ln \frac{1-\tilde{\beta}}{1+\tilde{\beta}} = \frac{1}{2\pi} \ln \frac{\tilde{\mu}_1 + \tilde{\mu}_2 \tilde{\kappa}_1}{\tilde{\mu}_2 + \tilde{\mu}_1 \tilde{\kappa}_2} \tag{9}$$

By consecutively substituting the function $\hat{F}(z)$ into Eqs. (7) and (6), the stresses and displacements in the transformed field are obtained. After taking the Laplace inverse transform, the stresses and displacements in the real time field are subsequently obtained.

5. Crack line and tip fields

The field solutions along the crack line (interface, $y = 0$) are generally of special interest. The traction along the interface can be obtained as,

$$\hat{\mathbf{t}}(x) = \hat{\sigma}_{yy} + \mathrm{i}\hat{\sigma}_{xy} = \frac{\hat{\sigma}_{yy}^{\infty} + \mathrm{i}\hat{\sigma}_{xy}^{\infty}}{\pm\sqrt{x^2 - a^2}} \left(\frac{x-a}{x+a}\right)^{\mathrm{i}\tilde{\varepsilon}} (x + 2\mathrm{i}\tilde{\varepsilon}a), \quad |x| > a \tag{10}$$

where signs + and − correspond to $x > a$ and $x < -a$, respectively. And the relative crack face displacements are,

$$\Delta\hat{\mathbf{U}}(x) = \Delta\hat{u}_y + \mathrm{i}\Delta\hat{u}_x = \frac{4}{\tilde{E}^*}(\hat{\sigma}_{yy}^{\infty} + \mathrm{i}\hat{\sigma}_{xy}^{\infty})\sqrt{a^2 - x^2}\left(\frac{a-x}{a+x}\right)^{\mathrm{i}\tilde{\varepsilon}} \Big/ \cosh(\pi\tilde{\varepsilon}), \quad |x| \leqslant a \tag{11}$$

where $1/\tilde{E}^* = (1/2)(1/\tilde{E}_1' + 1/\tilde{E}_2')$, $1/\tilde{E}_i' = 1/\tilde{E}_i$ for plane stress and $(1-\tilde{v}_i^2)/\tilde{E}_i$ for plane strain and $\tilde{E}_i = 9\tilde{\mu}_i\tilde{K}_i/(3\tilde{K}_i + \tilde{\mu}_i)$, $(i = 1, 2)$.

The crack tip fields are of particular importance, and we will focus on one crack tip, such as the right tip, and for the other tip, the fields are similar. The traction in the interface at a distance r ahead of the (right) crack tip ($\theta = 0°$ in the local crack tip coordinate system) can be obtained from Eq. (10) as,

$$\hat{\mathbf{t}}(r) = \hat{\sigma}_{yy} + \mathrm{i}\hat{\sigma}_{xy} = \frac{\hat{K}}{\sqrt{2\pi r}}\left(\frac{r}{2a}\right)^{\mathrm{i}\tilde{\varepsilon}}(1 + 2\mathrm{i}\tilde{\varepsilon}) \tag{12}$$

where $\hat{K} = (\hat{\sigma}_{yy}^{\infty} + \mathrm{i}\hat{\sigma}_{xy}^{\infty})\sqrt{\pi a}$ for a finite crack under remote uniform loading. From Eq. (11), the relative crack face displacements at a distance r behind the (right) crack tip ($\theta = 180°$ in the crack tip system), are

$$\Delta\hat{\mathbf{U}}(r) = \Delta\hat{u}_y + \mathrm{i}\Delta\hat{u}_x = \frac{8}{\cosh(\pi\tilde{\varepsilon})}\frac{\hat{K}}{\tilde{E}^*}\sqrt{\frac{r}{2\pi}}\left(\frac{r}{2a}\right)^{\mathrm{i}\tilde{\varepsilon}} \tag{13}$$

After taking the inverse Laplace transformation of Eqs. (12) and (13), the crack tip traction and displacement fields can be obtained. The energy release rate for a unit area of interface to decohere can be determined from the expressions for the traction and displacement fields by the closure integral,

$$G = \frac{1}{2\delta}\int_0^{\delta} \bar{\mathbf{t}}(\delta - r)\Delta\mathbf{U}(r)\,\mathrm{d}r \tag{14}$$

where δ is an arbitrary length scale.

5.1. An approximate method

As mentioned earlier, the full crack fields in the transformed field are obtained easily, however, to obtain the solutions in the real time field requires performing Laplace inverse transform. Generally, Laplace inverse transform cannot be obtained analytically, and one must resort to either numerical, or some kind of approximate methods. With respect to approximate methods, there is one in the viscoelastic response which merits special attention. It is Schapery's direct inverse method (Schapery 1961, 1967). In the following, first we will give a brief overview of the Schapery's direct Laplace inverse technique, and the "quasi-elastic" analysis method derived from it. Then, we will propose another approximate method which is more suitable for the problem at hand.

The Schapery's direct method for the approximate Laplace inversion of a given function, say $f(t)$, is

$$f(t) \approx p\hat{f}(p)|_{p=\gamma/t} \tag{15}$$

where $\gamma = e^{-C} \approx 1/2$ (C is Euler's constant). This equation is appropriate if the plot of $f(t)$ or $p\hat{f}(p)$ versus $\log t$ ($\log = \log_{10}$) has small curvature, and is exact when $f(t)$ is proportional to $\log t$.

The approximate method described below is derived from the direct inverse method and the definition of equivalent moduli of viscoelastic materials.

Suppose that the solution (response) of an elastic body with moduli C (μ, v, E, K, etc.) under constant loading condition (input) σ, is $\sigma f_e(C)$. Then according to the correspondence principle, under input $\sigma H(t)$ (where $H(t)$ is the Heaviside unit step function), the response of an viscoelastic body with equivalent moduli $\tilde{C}$ ($\tilde{\mu}$, $\tilde{v}$, $\tilde{E}$, $\tilde{K}$, etc.), in the Laplace field, will be $\hat{f}(p) = \sigma f_e(\tilde{C})/p$, and thus $p\hat{f}(p) = \sigma f_e(\tilde{C})$. Substituting into Eq. (15), it turns out that $f(t) \approx \sigma f_e(\tilde{C})|_{p=\gamma/t}$. Recalling the definition of equivalent moduli, $\tilde{C} = p\hat{C}$ and using Eq. (15), we have $\tilde{C}|_{p=\gamma/t} \approx C(t)$. And finally, the response of the viscoelastic body can be approximated by,

$$f(t) \approx \sigma f_e[C(t)] \tag{16}$$

Thus, the quasi-elastic method described above implies that a viscoelastic solution is approximated by an elastic solution wherein all elastic constants are replaced by the corresponding time-dependent moduli.

From the crack line and tip Eqs. (10)–(13), it can be seen that these formulae are of the form,

$$\hat{f}(p) = \hat{\sigma} f_1(\tilde{C}) f_2(\tilde{\varepsilon}) \tag{17}$$

where the input $\hat{\sigma} = \hat{\sigma}_{yy}^{\infty} + i\hat{\sigma}_{xy}^{\infty}$, $f_2(\tilde{\varepsilon})$ is a function which depends on $\tilde{\varepsilon}$, and $f_1(\tilde{C})$ depends on other moduli, except $\tilde{\varepsilon}$ (such as on $\tilde{E}'$ for the displacements) or independent of any moduli (such as for the stresses).

The solution of Eq. (17) should be obtained by $f(t) = \mathscr{L}^{-1}[\hat{\sigma} f_1(\tilde{C}) f_2(\tilde{\varepsilon})]$. It seems impossible to take the Laplace inverse transform of Eq. (17) and obtain an exact solution of $f(t)$ analytically. If the quasi-elastic method is used, $f(t) \approx \sigma f_1[C(t)] f_2[\varepsilon(t)]$. Based on the direct Laplace inverse method and quasi-elastic method, we suggest an approximate Laplace inverse method as,

$$f(t) \approx \mathscr{L}^{-1}[\hat{\sigma} f_1(\tilde{C})] f_2[\varepsilon(t)] \tag{18}$$

The above implies that the approximate solution can be obtained in two steps: (a) take the Laplace inverse transform by considering $\tilde{\varepsilon}$ as a constant, and (b) replace $\tilde{\varepsilon}$ in the result of (a) by $\varepsilon(t)$. This method is similar to the quasi-elastic method applied only to the $\tilde{\varepsilon}$ influenced term, $f_2(\tilde{\varepsilon})$, i.e. $f_2(\tilde{\varepsilon})$ is replaced by $f_2[\varepsilon(t)]$.

We now give an illustrative explanation regarding the approximate method. If $f_1(\tilde{C})$ is independent of any moduli, that is $\hat{f}(p)$ depends only on $\tilde{\varepsilon}$, as in the case of the stress equations (10) and (12), the new approximate method is then equivalent to the quasi-elastic method. If $\tilde{\varepsilon}$ is constant, and so is $f_2(\tilde{\varepsilon})$, then the proposed method (18) is an exact one, while the quasi-elastic method, $f(t) \approx \sigma f_1[C(t)] f_2[\varepsilon(t)]$, may not be exact.

Before considering $\tilde{\varepsilon}$ and $f_2(\tilde{\varepsilon})$ further, let us examine ε in an elastic bi-material system. The parameter ε (the oscillatory singularity index is equal to $(1/2\pi)\ln((\mu_1 + \mu_2\kappa_1)/(\mu_2 + \mu_1\kappa_2))$ and its effect have been discussed extensively in the elastic interface fracture mechanics (Hutchinson and Suo, 1991). The physically admissible value of ε is restricted to within $|\varepsilon| \leqslant \ln 3/(2\pi) \approx 0.175$. In reality, for typical bi-material systems, ε is very small, and its effect on fracture is often of secondary importance. As an approximation, occasionally the effect of ε is neglected. In the viscoelastic problem, usually $\tilde{\varepsilon}$ is quite small (in fact, $|\tilde{\varepsilon}| \leqslant \max|\varepsilon(t)|$), the variation of $\tilde{\varepsilon}$ and its influence term $f_2(\tilde{\varepsilon})$ are smooth functions of time (which satisfy the small curvature condition of the direct inverse method). Thus, by using the proposed method, which introduces approximation only in the $\tilde{\varepsilon}$ influence term, one would obtain results with sufficient accuracy.

In the calculation of $\varepsilon(t)$, instead of using definition (9), an approximate expression in which the elastic constants are replaced by corresponding time-dependent moduli, is used,

$$\varepsilon(t) = \frac{1}{2\pi} \ln \frac{\mu_1(t) + \mu_2(t)\kappa_1(t)}{\mu_2(t) + \mu_1(t)\kappa_2(t)} \tag{19}$$

where

$$\kappa_i(t) = \begin{cases} 3 - 4v_i(t), & \text{for plane strain} \\ [3 - v_i(t)]/[1 + v_i(t)], & \text{for plane stress} \end{cases}, \quad v_i(t) = \frac{3K_i(t) - 2\mu_i(t)}{2[3K_i(t) + \mu_i(t)]}, \quad i = 1, 2.$$

This is equivalent to applying the direct inverse method to the definition of $\tilde{\varepsilon}$ (including $\tilde{\kappa}$ and $\tilde{v}$).

5.2. Approximate crack line and tip fields

In the following, we apply the approximate Laplace inverse transform method to Eqs. (10)–(13), to obtain analytical formulae of the stress and displacement fields on the crack line and at the crack tip.

From Eq. (10), the traction along the interface can be obtained (approximately) as,

$$\mathbf{t}(x,t) = \sigma_{yy} + i\sigma_{xy} = \frac{\sigma_{yy}^{\infty}(t) + i\sigma_{xy}^{\infty}(t)}{\pm\sqrt{x^2 - a^2}} \left(\frac{x-a}{x+a}\right)^{i\varepsilon(t)} [x + 2ia\varepsilon(t)], \quad |x| > a \tag{20}$$

where signs + and − correspond to $x > a$ and $x < -a$, respectively.

From Eq. (11), the relative crack face displacements are

$$\begin{aligned} \Delta\mathbf{U}(x,t) &= \Delta u_y + i\Delta u_x \\ &= \left\{ \mathscr{L}^{-1}\left(\frac{1}{\tilde{E}^*}\right) * [\sigma_{yy}^{\infty}(t) + i\sigma_{xy}^{\infty}(t)] \right\} 4\sqrt{a^2 - x^2} \left(\frac{a-x}{a+x}\right)^{i\varepsilon(t)} \Bigg/ \cosh[\pi\varepsilon(t)], \quad |x| \leqslant a \end{aligned} \tag{21}$$

The operational symbol $*$ expresses convolution integration, that is

$$f_1(t) * f_2(t) = \int_{-\infty}^{\infty} f_1(\tau) f_2(t-\tau)\, d\tau$$

The crack tip stresses can be obtained from Eq. (12),

$$\mathbf{t}(r,t) = \sigma_{yy} + i\sigma_{xy} = \frac{K(t)}{\sqrt{2\pi r}} \left(\frac{r}{2a}\right)^{i\varepsilon(t)} [1 + 2i\varepsilon(t)] \tag{22}$$

In the case of a finite crack, $K(t) = [\sigma_{yy}^{\infty}(t) + i\sigma_{xy}^{\infty}(t)]\sqrt{\pi a}$.

The relative crack face displacements determined from Eq. (13) is,

$$\Delta\mathbf{U}(r,t) = \Delta u_y + i\Delta u_x = \frac{8}{\cosh[\pi\varepsilon(t)]} \sqrt{\frac{r}{2\pi}} \left(\frac{r}{2a}\right)^{i\varepsilon(t)} \left[K(t) * \mathscr{L}^{-1}\left(\frac{1}{\tilde{E}^*}\right)\right] \tag{23}$$

Substituting Eqs. (22) and (23) into Eq. (12), the energy release rate for a unit area of interface to decohere is obtained in the form of:

$$G(t) = \frac{1 + 4[\varepsilon(t)]^2}{\cosh^2[\pi\varepsilon(t)]} \overline{K(t)} \left[K(t) * \mathscr{L}^{-1}\left(\frac{1}{\tilde{E}^*}\right)\right] \tag{24}$$

For a proportional loading condition, that is $\sigma_{xy}^{\infty}(t)=h\sigma_{yy}^{\infty}(t)$ and denoting $\sigma_{yy}^{\infty}(t)=\sigma_{yy}^{\infty}f_\sigma(t)$, then $K(t)=Kf_\sigma(t)$, with $K=\sigma_{yy}^{\infty}(1+\mathrm{i}h)\sqrt{\pi a}$. Eqs. (22)–(24) can be expressed further as,

$$\mathbf{t}(r,t)=\sigma_{yy}+\mathrm{i}\sigma_{xy}=\frac{K}{\sqrt{2\pi r}}\left(\frac{r}{2a}\right)^{\mathrm{i}\varepsilon(t)}[1+2\mathrm{i}\varepsilon(t)]f_\sigma(t) \tag{25}$$

$$\Delta\mathbf{U}(r,t)=\Delta u_y+\mathrm{i}\Delta u_x=\frac{8K}{E^*\cosh[\pi\varepsilon(t)]}\sqrt{\frac{r}{2\pi}}\left(\frac{r}{2a}\right)^{\mathrm{i}\varepsilon(t)}f_u(t) \tag{26}$$

$$G(t)=\frac{|K|^2}{E^*}\frac{1+4[\varepsilon(t)]^2}{\cosh^2[\pi\varepsilon(t)]}f_G(t) \tag{27}$$

with

$$f_G(t)=f_\sigma(t)f_u(t),\qquad f_u(t)=f_\sigma(t)*\mathscr{L}^{-1}\left(\frac{E^*}{\tilde{E}^*}\right),\qquad \frac{1}{E^*}=\frac{1}{2}\left[\frac{1}{E_1'(0^+)}+\frac{1}{E_2'(0^+)}\right]$$

If the small variation due to $\varepsilon(t)$ is ignored, it can be seen from Eqs. (25)–(27) that $\mathbf{t}(x,t)/\mathbf{t}(x,0^+)\approx f_\sigma(t)$, $\Delta\mathbf{U}(x,t)/\Delta\mathbf{U}(x,0^+)\approx f_u(t)$ and $G(t)/G(0^+)\approx f_G(t)=f_\sigma(t)f_u(t)$. The functions $f_\sigma(t)$, $f_u(t)$ and $f_G(t)$ are called time indices of stress, displacement and energy release rate, respectively.

Introducing the extension relaxation compliance $C_i(t)=\mathscr{L}^{-1}(1/(p\tilde{E}_i'))$ $(i=1,2)$, and recalling $1/\tilde{E}^*=(1/2)(1/\tilde{E}_1'+1/\tilde{E}_2')$, then $\mathscr{L}^{-1}(1/\tilde{E}^*)=(1/2)\mathrm{d}/\mathrm{d}t[C_1(t)+C_2(t)]$ and the time index $f_u(t)$ can be expressed as,

$$f_u(t)=f_\sigma(t)*\frac{\mathrm{d}}{\mathrm{d}t}[C_1(t)+C_2(t)]/[C_1(0^+)+C_2(0^+)] \tag{28}$$

For the special case of $f_\sigma(t)=H(t)$ (with $H(t)$ denoting the Heaviside unit step function),

$$f_u(t)=[C_1(t)+C_2(t)]/[C_1(0^+)+C_2(0^+)]\quad\text{and}\quad f_G(t)=f_u(t).$$

6. Initial and terminal states

At $t=0$ and $t\to\infty$ the crack fields can be determined simply without Laplace inverse transform, according to the theory,

$$f(0^+)=\lim_{t\to0^+}f(t)=\lim_{p\to\infty}p\hat{f}(p),\qquad f(\infty)=\lim_{t\to\infty}f(t)=\lim_{p\to0}p\hat{f}(p)$$

and we will denote $f(0^+)$ as f_0 and $f(\infty)$ as f_∞ for the sake of abbreviation.

At the initial time $t=0^+$, the crack tip traction and crack surface displacement jump can be obtained directly from Eqs. (12) and (13) as,

$$\mathbf{t}_0=\frac{K_0}{\sqrt{2\pi r}}\left(\frac{r}{2a}\right)^{\mathrm{i}\varepsilon_0}(1+2\mathrm{i}\varepsilon_0) \tag{29}$$

$$\Delta\mathbf{U}_0=\frac{8}{\cosh(\pi\varepsilon_0)}\frac{K_0}{E^*}\sqrt{\frac{r}{2\pi}}\left(\frac{r}{2a}\right)^{\mathrm{i}\varepsilon_0} \tag{30}$$

where $\lim_{p\to\infty}\tilde{\varepsilon}=\lim_{p\to\infty}p\hat{\varepsilon}(p)=\varepsilon_0$ and $\lim_{p\to\infty}\tilde{E}^*=E^*$ have been used. For the finite crack, under loading $\sigma_{yy}^{\infty}(t)=\sigma_{xy}^{\infty}(t)/h=\sigma_{yy}^{\infty}f_\sigma(t)$, we have $K_0=\sigma_{yy}^{\infty}(1+\mathrm{i}h)\sqrt{\pi a}f_\sigma(0)=Kf_\sigma(0)$. From Eq. (14), the initial energy release rate is,

$$G_0 = \frac{|K_0|^2}{E^*}\frac{1+4\varepsilon_0^2}{\cosh^2(\pi\varepsilon_0)} \tag{31}$$

Similarly, at the terminal time $t \to \infty$,

$$\mathbf{t}_\infty = \frac{K_\infty}{\sqrt{2\pi r}}\left(\frac{r}{2a}\right)^{\mathrm{i}\varepsilon_\infty}(1+2\mathrm{i}\varepsilon_\infty) \tag{32}$$

$$\Delta\mathbf{U}_\infty = \frac{8}{\cosh(\pi\varepsilon_\infty)}\frac{K_\infty}{E_\infty^*}\sqrt{\frac{r}{2\pi}}\left(\frac{r}{2a}\right)^{\mathrm{i}\varepsilon_\infty} \tag{33}$$

$$G_\infty = \frac{|K_\infty|^2}{E_\infty^*}\frac{1+4\varepsilon_\infty^2}{\cosh^2(\pi\varepsilon_\infty)} \tag{34}$$

with $\lim_{p\to 0}\tilde{\varepsilon} = \lim_{p\to 0} p\hat{\varepsilon}(p) = \varepsilon_\infty$, $\lim_{p\to 0} 1/\tilde{E}^* = 1/E_\infty^*$ and $K_\infty = \sigma_{yy}^\infty(1+\mathrm{i}h)\sqrt{\pi a} f_\sigma(\infty) = K f_\sigma(\infty)$.

Note that the traction, displacements and energy release rate along the crack line at $t = 0^+$ and $t \to \infty$, which were obtained by using the described approximate method, are exact.

7. Anti-plane field

For the plane problem, anti-plane and in-plane deformations are decoupled; thus, they can be treated separately. The anti-plane field can be represented by one complex potential $\omega(z)$. In the Laplace transformed field,

$$\hat{\sigma}_{yz} = \mathrm{Re}[\hat{\omega}'(z)], \qquad \tilde{\mu}\hat{u}_z = \mathrm{Im}[\hat{\omega}(z)] \tag{35}$$

For a finite interface crack of length $2a$, under remote uniform anti-plane shear stress $\sigma_{yz}^\infty(t)$, the complex potential is given by

$$\hat{\omega}'(z) = \hat{\sigma}_{yz}^\infty \frac{z}{\sqrt{z^2-a^2}} \tag{36}$$

Substituting into Eq. (35), and taking the Laplace inverse transform, the full anti-plane fields can then be obtained.

The stress at a distance r ahead of the crack tip is,

$$\sigma_{yz}(t) = \frac{K_{\mathrm{III}}}{\sqrt{2\pi r}} f_{\mathrm{III}\sigma}(t) \tag{37}$$

where K_{III} and $f_{\mathrm{III}\sigma}(t)$ are the anti-plane stress intensity factor and the time index of stress. In the case of a finite crack, under remote uniform loading $\sigma_{yz}^\infty(t) = \sigma_{yz}^\infty f_{\mathrm{III}\sigma}(t)$ and $K_{\mathrm{III}} = \sigma_{yz}^\infty\sqrt{\pi a}$.

The displacement jump at a distance r behind the crack tip is,

$$\Delta u_z(t) = \frac{K_{\mathrm{III}}}{2\mu^*}\sqrt{\frac{r}{2\pi}} f_{\mathrm{III}\sigma}(t) * \mathscr{L}^{-1}\frac{\mu^*}{\tilde{\mu}^*} \tag{38}$$

with

$$\frac{1}{\tilde{\mu}^*} = \frac{1}{2}\left(\frac{1}{\tilde{\mu}_1}+\frac{1}{\tilde{\mu}_2}\right) \quad \text{and} \quad \frac{1}{\mu^*} = \frac{1}{2}\left[\frac{1}{\mu_1(0^+)}+\frac{1}{\mu_2(0^+)}\right]$$

Denoting the time index $f_{\mathrm{III}u}(t) = f_{\mathrm{III}\sigma}(t) * \mathscr{L}^{-1}(\mu^*/\tilde{\mu}^*)$, Eq. (38) can be expressed as,

$$\Delta u_z(t) = \frac{K_{\mathrm{III}}}{2\mu^*}\sqrt{\frac{r}{2\pi}} f_{\mathrm{III}u}(t) \tag{39}$$

Alternatively, introducing the displacement relaxation compliance $J(t) = \mathscr{L}^{-1}(1/p2\tilde{\mu})$, the time index $f_{\mathrm{III}u}(t)$ is given by,

$$f_{\mathrm{III}u}(t) = f_{\mathrm{III}\sigma}(t) * \frac{\mathrm{d}}{\mathrm{d}t}[J_1(t) + J_2(t)]/[J_1(0^+) + J_2(0^+)] \tag{40}$$

The energy release rate can be obtained from the expressions for the traction and displacement fields, as

$$G_{\mathrm{III}}(t) = G_{\mathrm{III}}(0) f_{\mathrm{III}G}(t), \qquad f_{\mathrm{III}G}(t) = f_{\mathrm{III}\sigma}(t) f_{\mathrm{III}u}(t) \tag{41}$$

where $G_{\mathrm{III}}(0) = K_{\mathrm{III}}^2/(2\mu^*)$ is the initial energy release rate, and $f_{\mathrm{III}G}(t)$ is time index of energy release rate.

In contrast to the in-plane problem, the anti-plane one is quite simple, and the exact expressions for the crack tip fields and fracture parameters can be obtained without recourse to the approximate Laplace inverse method.

8. Fracture criterion along the interface

The criterion for the initiation of crack advance along the interface can be based on the energy release rate reaching the toughness of the interface, that is,

$$G(t) = \Gamma \tag{42}$$

The energy release rate should include $G_{\mathrm{III}}(t)$ if there exists an anti-plane loading. The interface toughness Γ is a material parameter which usually depends on the loading mode. In general, it is determined experimentally. Simple type of interface toughness functions are generally adopted for the in-plane problems, e.g. see Hutchinson and Suo (1991),

$$\Gamma(\psi) = G_{1\mathrm{c}}[1 + (1-\lambda)\tan^2\psi] \tag{43}$$

where $G_{1\mathrm{c}}$ is the pure mode I toughness, $\psi = \tan^{-1}(\sigma_{xy}^{\infty}/\sigma_{yy}^{\infty})$ represents the mixed loading mode, the parameter $0 \leqslant \lambda \leqslant 1$ adjusts the influence of the mode II contribution in the criterion.

When the energy release rate of a crack varies with time, the crack advance criterion (42) may be reached at any time during loading history:

Case A: $G(0) \geqslant \Gamma$, the crack advances instantaneously upon the load application.

Case B: $G(0) \leqslant \Gamma$ and $G(t_{\mathrm{c}}) = \Gamma$, the crack advances after time t_{c} of loading.

Case C: $G(0) \leqslant \Gamma$ and $G(\infty) < \Gamma$, the crack does not propagate.

The condition $G(0) = \Gamma$ is the critical state of instantaneous crack propagation, whereas $G(\infty) = \Gamma$ is the critical state of retardation of the crack propagation.

9. Other viscoelastic effects

In this paper our main focus is the interfacial fracture; hence, attention is paid on the viscoelastic fields and energy release rate along the cracked interfacial plane. In the following, the crack fields at any point of the viscoelastic plane, and the interfacial crack branching possibilities will be discussed.

It can be seen from Eqs. (6)–(8), that the stress fields do not depend on any other material moduli except the bi-material oscillatory singularity index $\tilde{\varepsilon}$ and the elastic mismatch parameter $\tilde{\beta}$. This implies that, except the small influence due to the variation of $\varepsilon(t)$, the viscoelastic stress fields are similar to the corresponding elastic fields. For example, under a constant remote loading, the stress fields in the viscoelastic material are also approximately constant. The displacements (or strains), however, depend mainly on the material moduli $\tilde{\mu}$ (and $\tilde{E}'$), approximately $\{\hat{u}_x, \hat{u}_y\} \propto 1/\tilde{\mu}$. It indicates that, under a constant remote loading, the change in displacements with time has a similar tend as the creep compliance $J(t)$ or $C(t)$ of the viscoelastic material.

There are two possibility for an interfacial crack to advance: (a) to remain in the interface (which has been analyzed above), or (b) to kink out of the interface. We will now investigate the crack kinking phenomenon due to the viscoelastic effect. Let us first review the results of the corresponding elastic problem studied by He and Hutchinson (1989) and He et al. (1991). The competition between interface cracking and substrate cracking (kinking) depends on whether the energy release ratio, $G_\mathrm{i}/G_\mathrm{s}^{\max}$ is greater or less than the toughness ratio, $\Gamma_\mathrm{i}/\Gamma_\mathrm{s}$. Here G_i is the energy release rate for the crack advancing in the interface and $G_\mathrm{s}^{\max}$ is the energy release rate for the crack kinking into the substrate which is maximized with respect to the kinking angle. The Γ_i (or $\Gamma_\mathrm{i}(\psi)$) is the interface toughness (or interface toughness function which depends on the mixed mode loading parameter ψ and Γ_s is the substrate toughness. If

$$G_\mathrm{i}/G_\mathrm{s}^{\max} > \Gamma_\mathrm{i}/\Gamma_\mathrm{s} \tag{44}$$

the crack will continue to advance along the interface. Conversely, if the inequality in Eq. (44) is reversed, the crack will kink into the substrate.

For viscoelastic bi-materials, the toughnesses of the interface and the substrate material are assumed not to change with time, that is, Γ_i, Γ_s and $\Gamma_\mathrm{i}/\Gamma_\mathrm{s}$ are constants. But the energy release rates of the interfacial cracking and substrate kinking usually will change with time and the rates of their changes are different, i.e. $G_\mathrm{i}(t)/G_\mathrm{s}^{\max}(t) \neq G_\mathrm{i}(0)/G_\mathrm{s}^{\max}(0)$. Here we will only focus on the changing trends of the interfacial crack energy release rate and the kinked crack energy release rate, and estimate the variation.

It has been shown earlier that, for the interface crack $G_\mathrm{i}(t)/G_\mathrm{i}(0) \approx f_\sigma(t) f_{u\mathrm{i}}(t)$, where $f_\sigma(t) = \sigma^\infty(t)/\sigma^\infty(0^+)$ is the time index of the remote loading, and approximately the time index of the relative interfacial displacement, $f_{u\mathrm{i}}(t) = f_\sigma(t) * (\mathrm{d}/\mathrm{d}t)[C_1(t) + C_2(t)]/[C_1(0) + C_2(0)]$ for the plane problem.

For the kinked crack, from the results of corresponding elastic problem, it is known that the stress field at the kinked crack tip is only weakly dependent on the material mismatch moduli. Therefore, it is reasonable to assume that the stress field at the kinked crack field in a viscoelastic bi-material system, is approximately independent of the material moduli. This then implies that the time index of the stress field is approximately equal to $f_\sigma(t)$ (the time index of remote loads). For the kinked crack tip located in the homogeneous material, such as kinking into material 1 (Fig. 1), the relative crack surface displacements along the kinked crack are proportional to $1/\tilde{E}_1'$ in the Laplace field, and its time index is approximately $f_{us}(t) = f_\sigma(t) * (\mathrm{d}/\mathrm{d}t)C_1(t)/C_1(0)$ for the plane problem. Then approximately, the time index of the energy release rate along kinking crack is $G_\mathrm{s}(t)/G_\mathrm{s}(0) \approx f_\sigma(t) f_{us}(t)$.

When the compliance of one material such as material 1, is higher than that of the other material, i.e. $C_1(t) > C_2(t)$, then $f_{u\mathrm{i}}(t) > f_{us}(t)$ and $G_\mathrm{s}(t)/G_\mathrm{s}(0)$ including $G_\mathrm{s}^{\max}(t)/G_\mathrm{s}^{\max}(0)$, will be greater than $G_\mathrm{i}(t)/G_\mathrm{i}(0)$, or $G_\mathrm{i}(t)/G_\mathrm{s}^{\max}(t) < G_\mathrm{i}(0)/G_\mathrm{s}^{\max}(0)$. The decreasing trend of the ratio $G_\mathrm{i}(t)/G_\mathrm{s}^{\max}(t)$ with time indicates the possibility of $G_\mathrm{i}/G_\mathrm{s}^{\max} < \Gamma_\mathrm{i}/\Gamma_\mathrm{s}$ increasing with time. Therefore, according to the criterion (44), the tendency for the crack to kink into the higher compliance (softer) substrate layer will increase with time. This is in keeping with the result of the corresponding elastic problem, i.e. increasing the relative compliance of the material into which the crack kinks, increases the energy release rate of the kinked crack, and thus increases the tendency for the substrate cracking (He and Hutchinson, 1989; He et al., 1991).

10. Example and results

The viscoelastic behavior of materials can be represented by a combination of elastic (springs) and viscous (dashpots) elements. In this example, the viscoelastic material behavior is modeled by the standard linear solid (see Fig. 2). The material stress–strain relation and parameters are given in the appendix.

Let us now consider a viscoelastic and elastic interface crack problem as an example. It is assumed that material 1 is an epoxy polymer with the following properties at the initial time $t = 0$

$$E_0 = 3.4 \text{ GPa}, \qquad \nu_0 = 0.3$$

At the initial time, other material parameters such as the shear and bulk moduli, can be calculated in terms of E_0 and ν_0, e.g. $\mu_0 = E_0/[2(1+\nu_0)]$ and $K_0 = E_0/[3(1-2\nu_0)]$. The epoxy has a viscoelastic behavior with its shear modulus represented by the standard linear solid (Fig. 2). It is assumed in this example that $\mu_\infty = \mu_0/10$ and the relaxation time of the shear modulus is τ_G. Then, the three parameters G_1, G_2 and η_2 of the standard linear solid viscoelastic model (see Fig. 2) can be determined in terms of μ_0, μ_∞ and τ_G (their relationships are given in the appendix). Besides the shear modulus, the behavior of another modulus, such as the bulk modulus, has to be specified. For the epoxy polymer, it is assumed that the bulk modulus K is constant, i.e. $K = K_0$. Thus, the viscoelastic material model is now completely specified (see Appendix A for the material parameters). Fig. 3 shows the material properties, including (both shear and extension) relaxation moduli and creep compliances of the epoxy polymer described above.

It is assumed that the material 2 is elastic, e.g. glass, with the following elastic constants:

$$E = 85 \text{ GPa}, \qquad \nu = 0.2$$

All of the other moduli of the glass can be determined accordingly, and they are also constant.

In the numerical calculations, the crack (a finite one with half length a) is assumed in plane strain condition, under quasi-statically applied remote uniform tension loading,

$$\sigma_{yy}^{\infty}(t) = \sigma_{yy}^{\infty} H(t)$$

with $H(t)$ denoting the Heaviside unit step function, the load being applied at time 0^-.

The tractions along the interface and near the crack tip are shown in Figs. 4 and 5. It can be seen that under the constant tension loading, the tractions along crack line are mainly tensile, and the tensile traction nearly does not change with time. The shear traction, which is quite small compared to the tension one, changes slightly with time only near the crack tip (such as $r/a < 0.05$) and tends to a terminal value after few times shear modulus retardation time, τ_G (in this example $\tau_G = 0.1\tau$, where τ is the shear compliance relaxation time). The tractions along crack line calculated by the approximate method, using Eq. (20) or

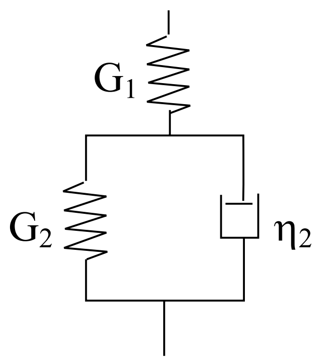

Fig. 2. Standard linear solid viscoelastic model.

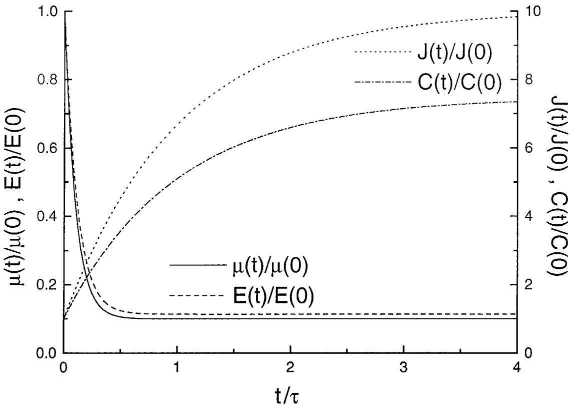

Fig. 3. Viscoelastic material properties, including relaxation moduli and creep compliances, in terms of dimensionless time, normalized by shear compliance retardation time τ.

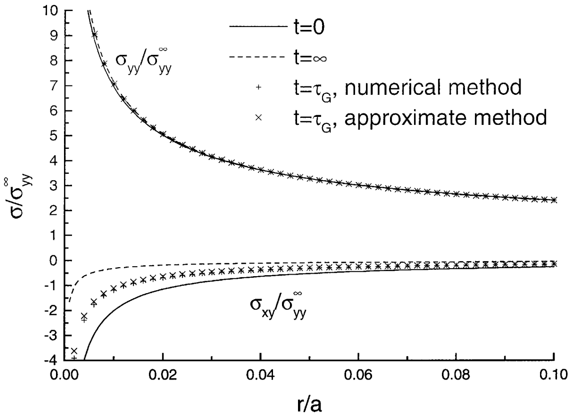

Fig. 4. Variation of tractions along the interface (ahead of the crack tip) with crack tip distance r, at time $t = 0$, τ_G (shear modulus relaxation time) and ∞, respectively.

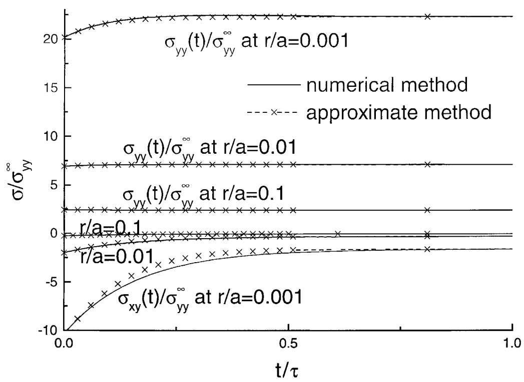

Fig. 5. Tractions along the interface near the crack tip $r/a = 0.1$, 0.01 and 0.001 respectively, as a function of normalized time t/τ.

(22), are also depicted in Figs. 4 and 5. It can be seen that the predicted values are very close to those calculated by the numerical Laplace inverse method (herein called the numerical method).

The relative crack surface displacements are shown in Figs. 6 and 7. It is seen that under a constant tensile loading, the relative displacements change significantly with time. Indeed, for the case that the moduli of the elastic material is quite greater than those of the viscoelastic material, the change of $\Delta u_y(t)$ is nearly proportional to the extension creep compliance $C(t)$ of the viscoelastic material (see Eq. (28)). Note that the shear crack face displacement Δu_x is quite small compared to the tensile one. Again, the crack face displacements calculated by the approximate method, using Eq. (21) (or (23) when $x \to a$), are quite accurate, they are hardly distinguishable from those by the direct numerical method. In Fig. 6, the quasi-elastic results are also shown. It is noted that they are not as accurate as the results by the approximate method proposed herein. In the quasi-elastic method for the displacements, the equivalent modulus $\tilde{E}'$ is replaced by the correspondent time-dependent function $E'(t) \approx 4\mu(t)((3K + \mu(t))/(3K + 4\mu(t)))$, whose

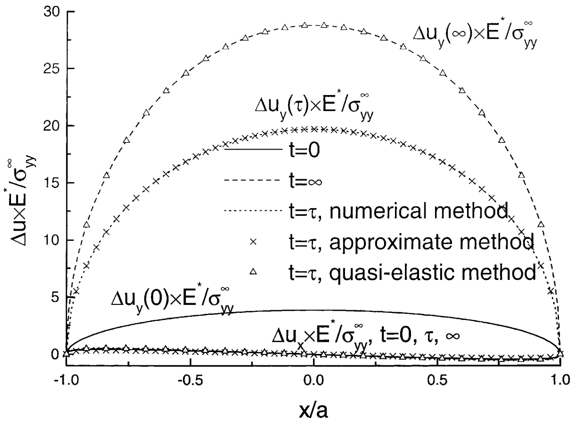

Fig. 6. Relative crack surface displacements, at times $t = 0$, τ and ∞, respectively.

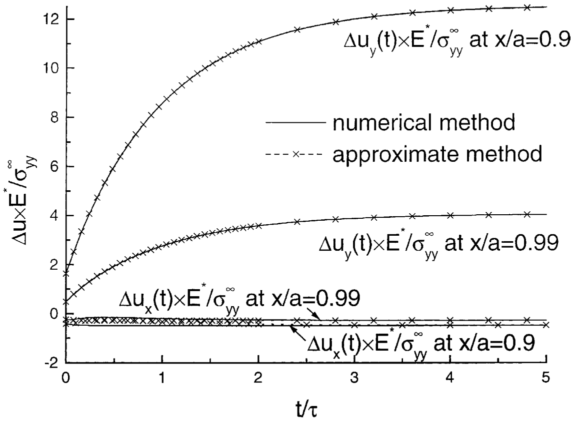

Fig. 7. Relative crack surface displacements at crack tip $x/a = 0.9$ and 0.99 respectively, as a function of normalized time t/τ.

relaxation time is quite shorter than the compliance retardation time τ, so that at time τ, the modulus $E'(t) \to E'(\infty)$, and the displacement results also tend to the terminal ones.

Fig. 8 shows the change of the oscillatory singularity index $\varepsilon(t)$, calculated both by its definition Eq. (9) and numerical inverse method, and by the approximate formula (19). It is seen that the approximate formula gives quite accurate results, and that the term $\varepsilon(t)$ changes smoothly with time.

Since the approximate formulae are quite accurate in the calculation of both the crack line tractions (including those at the crack tip) and the relative crack surface displacements, then, Eq. (24) can be expected to be accurate enough to calculate the interface crack energy release rate (along crack line). Fig. 9 shows the change of energy release rate (along interface crack line) with time, calculated both by the approximate method and numerical method. Again, the approximate method predicts quite accurate results. The change of energy release rate $G(t)$ is quite large. Its change is approximately proportional to $f_G(t) = f_\sigma(t)f_u(t)$. And in the case of a constant loading, $f_G(t) = f_u(t)$, which indicates that the change of $G(t)$ is mainly due to the change of $\Delta u_y(t)$, and thus, $G(t)$ has a similar trend as $\Delta u_y(t)$.

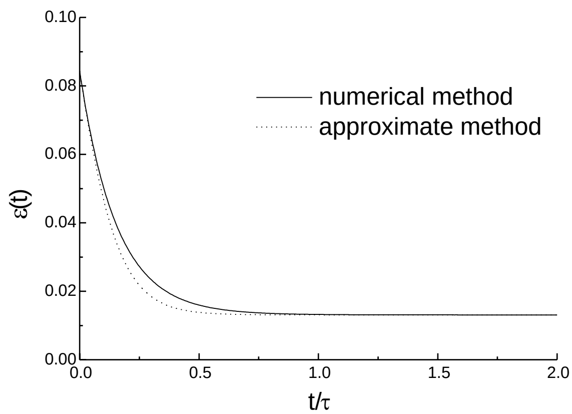

Fig. 8. The oscillatory singularity index $\varepsilon(t)$ changing with normalized time t/τ.

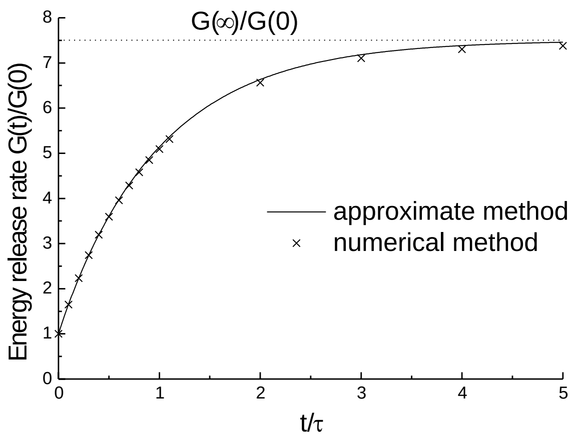

Fig. 9. The energy release rate of interface crack $G(t)$ changing with normalized time t/τ.

11. Conclusions

In this paper, the plane problem of an interfacial crack between different viscoelastic/elastic media is considered. By using the complex variable formulation, and the Laplace transformation, the viscoelastic crack fields are derived in the transformed field. Through an approximate Laplace inverse method, the interface crack line (and tip) fields and fracture parameters are obtained directly. As an illustrative example, an interfacial crack between a viscoelastic and an elastic media, is considered. It is shown that the results obtained by the approximate method are quite accurate compared to those determined by a direct numerical method.

Through the analytical formulae obtained by the approximate method, it can be seen that the change of energy release rate of an interface crack is approximately equal to the product of the change of loading and the change of relative crack displacements. When the applied load is constant, the change of energy release rate is approximately equal to the change of relative crack displacement. For a bi-material system in which the modulus of one material is quite higher than the other one, the change of crack surface displacements are mainly due to the change of the softer (more compliant) material, and nearly proportional to the extension creep compliance of the softer viscoelastic material. This also implies that the change of the energy release rate along the interface is mainly determined by the softer viscoelastic material.

The growth of an interfacial crack depends on the crack energy release rate and the interface toughness. Under constant or increasing load, the crack energy release rate will increase with time, and so will the possibility of the crack advance. An interfacial crack may also kink out of the interface, and the tendency for the crack kinking into the more compliant substrate will increase with time.

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Appendix A

A viscoelastic material described by the standard linear solid shown in Fig. 2 has the stress–strain relation,

$$(a_1D+1)s_{ij} = (b_1D+b_0)e_{ij}$$

with $a_1 = \eta_2/(G_1+G_2)$, $b_0 = 2G_1G_2/(G_1+G_2)$, $b_1/b_0 = \eta_2/G_2$.

The equivalent shear modulus is then given by $2\tilde{\mu} = (b_1p+b_0)/(a_1p+1) = 2G_1(\eta_2p+G_2)/(\eta_2p+G_1+G_2)$.

The shear relaxation modulus (shear stress under unit step shear strain) is,

$$G(t) = 2\mu(t) = 2G_1\left\{1 - \frac{G_1}{G_1+G_2}[1-\exp(-t/\tau_G)]\right\}$$

with $\mu_0 = G_1$, $\mu_\infty = G_1G_2/(G_1+G_2)$ and shear (modulus) relaxation time $\tau_G = \eta_2/(G_1+G_2)$. While the shear creep compliance (shear strain under unit step shear stress) is,

$$J(t) = \mathscr{L}^{-1}\frac{1}{p2\tilde{\mu}} = \frac{1}{2G_1} + \frac{1}{2G_2}[1-\exp(-t/\tau)]$$

with shear (compliance) retardation time $\tau = \eta_2/G_2$ and $\tau = (\mu_0/\mu_\infty)\tau_G$.

The extension relaxation modulus can be obtained from the relation $\tilde{E} = 9\tilde{K}\tilde{\mu}/(3\tilde{K} + \tilde{\mu})$. The extension creep compliance is given by the relation $\tilde{C} = 1/(p\tilde{E}')$ (with $\tilde{E}' = \tilde{E}$ for plane stress and $\tilde{E}/(1 - \tilde{v}^2)$ for plane strain).

For the case $v =$ constant:

$$E(t) = 2(1 + v)\mu(t), \qquad C(t) = \begin{cases} J(t)/(1 + v), & \text{for plane stress} \\ J(t)(1 - v), & \text{for plane strain} \end{cases}$$

For the case $K =$ constant:

$$E(t) = E_0 - (E_0 - E_\infty)[1 - \exp(-t/\tau_E)]$$

with $E_0 = 9K\mu_0/(3K + \mu_0)$, $E_\infty = 9K\mu_\infty/(3K + \mu_\infty)$ and the extension (modulus) relaxation time $\tau_E = (E_\infty/E_0)\tau$. For the plane stress,

$$C(t) = \frac{1}{E_0} + \left(\frac{1}{E_\infty} - \frac{1}{E_0}\right)[1 - \exp(-t/\tau)]$$

and for the plane strain,

$$C(t) = C_0\{1 + A_1[1 - \exp(-t/\tau)] + A_2[1 - \exp(-t/\tau_E)]\}$$

with

$$C_0 = \frac{1 - v_0^2}{E_0}, \quad A_1 = \left(\frac{1 + v_0}{1 - v_0}\frac{\mu_p - E_p}{1 - E_p} + 1\right)\frac{G_1}{G_2}, \quad A_2 = \frac{1 + v_0}{1 - v_0}\frac{(\mu_p - E_p)^2}{E_p(E_p - 1)}, \quad \mu_p = \frac{\mu_0}{\mu_\infty}, \quad E_p = \frac{E_0}{E_\infty}.$$

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